What are Continued Fractions?

I was reading a great book recently, and it got me thinking about continued fractions.[1] A continued fraction is something that looks like

$$n_0 + \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \cfrac{n_4}{d_4 + \cfrac{n_5}{\ddots}}}}}$$

where the diagonal dots mean the process goes on forever like a bottomless well. At this point we have no idea what kind of criteria the (n_i, d_i) pairs need to satisfy for this to come out to an actual number, but don't worry about that. We won't go into lots of detail about continued fractions; we'll just play with a very special one.

A Special Continued Fraction

Let's look at the following:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}$$

Since we have no idea what it is, we might as well give it a name; call it x. At the end of this note you'll find a `Python` program that will approximate x by going down the bottomless only a finite number of steps; you might be able to run it on your own computer, or there are a number of `Python` interpreters online—a few are listed in a comment above the program's main code. Don't run it yet, though! You'll ruin the surprise.

Now what?

I'm not an expert in continued fractions, so (perhaps like you) I don't know the first thing about dealing with them in general. I found something that will help us with this particular one, but more on that later.

I had an incredible math teacher in High school[2] who had a great attitude about tackling hard problems:

If you don't know what to do, do something.

[1] Yeah, yeah, sometimes I read really nerdy books.

[2] Ask Mr. Bill about Paul Machemer sometime.

The wisdom of continually trying things until something works can't be overstated; you're bound to get somewhere eventually. Even if you don't, you will end up knowing more about the problem than you did before, and that's just as important.

OK x, just what are you?

Let's look at x again:

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}$$

Remember that the dots mean that the fractions continue on forever and ever. If you look closely, you might notice something strange: the stuff under the first fraction bar is just x all over again! This means

$$x = 1 + \frac{1}{x}$$

with which I feel much more comfortable. Multiplying through by x to get rid of the denominators, we have

$$\begin{aligned} x^2 &= x + 1 \\ \implies x^2 - x - 1 &= 0 \end{aligned}$$

Using the handy-dandy quadratic formula[3], we have

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2} \\ &= \frac{1 + \sqrt{5}}{2} \text{ or } \frac{1 - \sqrt{5}}{2} \end{aligned}$$

Although we have two possible answers for x, only one makes sense. Note that the first one is positive, while the second one is negative (check it with your calculator, don't just take my word for it); there was no subtraction or negative numbers or any of that nonsense in our original continued fraction, so the negative value doesn't make sense as an answer.[4] That means that we have

$$x = \frac{1 + \sqrt{5}}{2}$$

also known as the φ, golden ratio! That thing pops up everywhere![5]

Happy Mathing!—*Mr. Bill's son Dr. E*

[3]Did you know that it fits to *Jingle Bells*? True story.

[4]There's room for a whole lot more detail here, but I'll skip it. If you're really interested, I have a short book (covered with my own notes and scribbles, sorry) that explores this much more in depth.

[5]If you don't believe me, check out "Donald Duck in Mathmagic Land," arguably the greatest movie of all time. I wouldn't be surprised if there was a copy up on YouTube, not that I endorse copyright infringement of course.

```python
# If you can't run Python programs yourself, try one of the following websites:
# http://repl.it/
# http://codepad.org/
# http://labs.codecademy.com/

def x(k):
    """Numerically investigate the continued fraction x =
    1 +        1
         ----------------
          1 +      1
               ----------
                1 +   1
                    -----
                      ...
    We won't be able to go out to infinity, so we'll use a counter to take only
    a (large) finite number of steps down this fraction.
    This is a recursive function; to find its answer, it will continue to call
    itself (with a different argument) until a certain stopping criterion is
    met.
    If our counter k is less than 1, we'll return 1; this is the stopping
    criterion---or base case---of our recursion.
    If not, we'll return 1 + 1 / x(k - 1) (our recursive step).
    """
    if k < 1:
        return 1.0
    else:
        return 1.0 + 1.0 / x(k - 1)


# Let's see x() in action
if __name__ == '__main__':
    # Print the outcome of stopping our continued fraction after 200 steps
    print(x(200)) # Does the output look familiar?
```